ON GENERALIZED CONVEX FUNCTIONS

BY

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1. **Introduction.** Let $\{F(x)\}$ be a family of real functions defined in the interval $a<x<b$ and satisfying the following conditions:

(1) Each $F(x)$ is a continuous function of x in $a<x<b$.

(2) There is a unique member of the family which, at arbitrary x_1, x_2 satisfying $a<x_1<x_2<b$, takes on arbitrary values y_1, y_2 respectively.

By $F(x; x_1, y_1; x_2, y_2)$ we shall denote the member of $\{F(x)\}$ which takes on values y_1, y_2 at x_1, x_2, respectively. For a given function $f(x)$ we shall for brevity denote $F[x; x_i, f(x_i); x_j, f(x_j)]$ by $F_{i,j}(x)$. We shall further denote the interval $a<x<b$ by (a, b).

DEFINITION. The real function $f(x)$, defined in (a, b), is a *sub-$F(x)$ function* provided the inequality

$$f\left(\frac{x_1+x_2}{2}\right) \leqq F_{1,2}\left(\frac{x_1+x_2}{2}\right) \tag{3}$$

holds for all x_1, x_2 satisfying $a<x_1<x_2<b$.

If the functions $F(x)$ are linear, then the class of sub-$F(x)$ functions is the class of convex functions. Since not all convex functions are continuous [4](1), it follows that (1), (2) and (3) do not imply that $f(x)$ is continuous. For a discussion of convex functions, see [2, 5, 6, 8].

We note that the above definition of a sub-$F(x)$ function is more general than that previously given [1], in which (3) was replaced by

$$f(x) \leqq F_{1,2}(x) \tag{4}$$

for all x_1, x_2, x satisfying $a<x_1<x<x_2<b$. Functions $f(x)$ satisfying (4) necessarily are continuous (Corollary 4, below); and for continuous functions $f(x)$, conditions (3) and (4) are equivalent (Lemma 3, below).

2. **The family $\{F(x)\}$.** We shall have need for the first two results given in this section. The third is included to show that for some families $\{F(x)\}$ satisfying (1) and (2) the class of sub-$F(x)$ functions is not topologically equivalent to the class of convex functions.

LEMMA 1 [1]. *For a given x_0 with $a<x_0<b$, let $F_r(x)$ and $F_s(x)$ be two members of a family $\{F(x)\}$ satisfying* (1) *and* (2), *such that*

$$F_r(x_0) = F_s(x_0), \tag{5}$$

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(1) Numbers in brackets refer to the references cited at the end of the paper.

(6) $$F_r(x) \not\equiv F_s(x) \qquad (a < x < b);$$

then $F_r(x) < F_s(x)$ *for all* x *in* (a, b) *on one side of* x_0, *while* $F_r(x) > F_s(x)$ *for all* x *in* (a, b) *on the other side of* x_0.

Proof. By (2), (5) and (6), $F_r(x) \neq F_s(x)$ in (a, b), except at x_0; consequently, by (1), on either side of x_0, one of $F_r(x)$, $F_s(x)$ is greater than the other. Suppose it could be the same one, say $F_s(x)$, which is greater on each side; we shall show that this is impossible.

Let x_1, x_2 satisfy $a < x_1 < x_0 < x_2 < b$, and consider the member $F_t(x)$ of $\{F(x)\}$, determined by $F_t(x) \equiv F[x; x_1, F_s(x_1); x_2, F_r(x_2)]$. We have $F_t(x_2) < F_s(x_2)$, so that $F_t(x) < F_s(x)$, $x_1 < x < b$; in particular,

(7) $$F_t(x_0) < F_s(x_0).$$

Similarly,

(8) $$F_t(x_0) > F_r(x_0).$$

Now (7) and (8) contradict (5).

LEMMA 2. *Let* x_0 *satisfy* $a < x_0 < b$ *and let* E *be a closed and bounded set of numbers in* (a, x_0) (*or in* (x_0, b)); *then for any real* y_0, M *there is a member* $F^*(x)$ *of* $\{F(x)\}$ *satisfying* $F^*(x_0) = y_0$ *and* $F^*(x) > M$ *for all* x *in* E.

Proof. For each x' in E there is an $h = h(x')$ such that if $I = I(x')$ denotes the interval $(x' - h, x' + h)$, we have

$$F(x; x_0, y_0; x', 2M) > M$$

for all x in I. By the Heine-Borel theorem there is a finite set $x_1', x_2', \cdots, x_n'$ of numbers in E such that each number in E is contained in at least one of $I(x_1'), I(x_2'), \cdots, I(x_n')$; therefore there is a finite set

$$R: \quad F_1(x), F_2(x), \cdots, F_n(x),$$

of members of $\{F(x)\}$ such that for each x' in E, we have

$$\max\,[F_1(x'), F_2(x'), \cdots, F_n(x')] > M.$$

By Lemma 1, some member of R satisfies the conditions specified for $F^*(x)$.

REMARK [7]. *Not all families* $\{F(x)\}$ *satisfying* (1) *and* (2) *are topologically equivalent to the family* $\{L(x)\}$ *of all nonvertical line segments terminating on* $x = a$, $x = b$.

Proof. Let the family $\{F(x)\}_0$ be determined for $-\infty < x < +\infty$ as follows: A function $F(x)$ is a member of $\{F(x)\}_0$ if and only if its graph in Cartesian coordinates is (i) a line of nonpositive slope or (ii) a continuous broken line, with break on the x-axis, such that the slope is positive for $y < 0$ and half as much for $y > 0$.

The family $\{F(x)\}_0$ satisfies conditions (1) and (2) in any interval (a, b).

Let ABC and $A'B'C'$ be triangles in the half-strip $a<x<b$, $y<0$, such that the lines AA', BB', CC' intersect in a point P of $a<x<b$, $y>0$; the slopes of AA' and BB' are negative while that of CC' is positive; and (see Desargues' Theorem [3]) the corresponding sides of the triangles intersect in collinear points in $a<x<b$, $y<0$.

Let $F_A(x)$, $F_B(x)$, $F_C(x)$ be the members of $\{F(x)\}_0$ whose graphs contain A, A'; B, B'; C, C', respectively. Then P is on $F_A(x)$ and on $F_B(x)$, but not on $F_C(x)$.

Suppose there were a reversible continuous transformation of the strip $a<x<b$ into itself carrying $\{F(x)\}_0$ into $\{L(x)\}$. Since a reversible continuous transformation preserves concurrency of graphs, triangles ABC and $A'B'C'$ are carried into triangles whose corresponding sides intersect in collinear points, so that, by the converse of Desargues' Theorem, the lines joining corresponding vertices are parallel or concurrent. This contradicts the fact that P is on $F_A(x)$ and on $F_B(x)$ but not on $F_C(x)$.

Actually we have shown more, that no plane domain D containing a segment of the x-axis can be mapped topologically on a plane domain D' in such a way that the graphs of the functions $\{F(x)\}_0$ in D are mapped on unbroken lines or line-segments in D'.

3. **Continuous sub-$F(x)$ functions.** We shall use the following lemma.

LEMMA 3. *If $f(x)$ is continuous in (a, b), then a necessary and sufficient condition that $f(x)$ be a sub-$F(x)$ function in (a, b) is that we have*

$$f(x) \leqq F_{1,2}(x) \tag{9}$$

for all x_1, x_2, x satisfying $a<x_1<x<x_2<b$.

Necessity. If (9) does not hold, then there exist x_3, x_4, x_5, with $a<x_3<x_5<x_4<b$, such that

$$f(x_5) > F_{3,4}(x_5). \tag{10}$$

By (10) and the continuity of $f(x)$ and of $F_{3,4}(x)$, there exist x_1, x_2, with $x_3 \leqq x_1 < x_5 < x_2 \leqq x_4$, such that

$$f(x) \geqq F_{3,4}(x) \qquad (x_1 \leqq x \leqq x_2), \tag{11}$$

the sign of equality holding at the end points but not elsewhere; then $F_{3,4}(x) \equiv F_{1,2}(x)$, so that (11) can be written as

$$f(x) > F_{1,2}(x) \qquad (x_1 < x < x_2). \tag{12}$$

Now (12) contradicts (3); hence if (9) does not hold, then $f(x)$ is not a sub-$F(x)$ function.

Sufficiency. If (9) holds, then (3) holds as a special case of (9), and consequently $f(x)$ is a sub-$F(x)$ function.

Our argument gives also the following result.

COROLLARY 1. *A necessary and sufficient condition that a function $f(x)$ continuous in (a, b) be a sub-$F(x)$ function is that for each pair of numbers x_1, x_2 satisfying $a<x_1<x_2<b$, there exist an x_3 satisfying $x_1<x_3<x_2$ such that $f(x_3) \leqq F_{1,2}(x_3)$.*

LEMMA 4. *A necessary and sufficient condition that a function $f(x)$ continuous in (a, b) be a sub-$F(x)$ function is that there exist a set E of numbers dense in (a, b) such that for each x_1, x_2 in E we have*

$$f\left(\frac{x_1 + x_2}{2}\right) \leqq F_{1,2}\left(\frac{x_1 + x_2}{2}\right). \tag{13}$$

Necessity. If $f(x)$ is a sub-$F(x)$ function, then the existence of a set E of numbers dense in (a, b) such that (13) holds for each x_1, x_2 in E follows from the definition of a sub-$F(x)$ function.

Sufficiency. As shown in Lemma 3, (12), if $f(x)$ is not a sub-$F(x)$ function, then there exist x_1, x_2 such that $f(x) > F_{1,2}(x)$ $(x_1 < x < x_2)$.

Let x_0 satisfy $a<x_0<x_1$ and let x_3 and x_4 divide (x_1, x_2) into three equal parts, $x_3<x_4$. For each x' such that $x_3 \leqq x' \leqq x_4$ there is a y' and an $h=h(x')$ such that if $I=I(x')$ is the interval $(x'-h, x'+h)$, then

$$f(x) > F[x;\, x_0, F_{1,2}(x_0);\, x', y'] > F_{1,2}(x)$$

for all x in I. A finite number of the intervals I covers (x_3, x_4). By Lemma 1, one of the corresponding functions $F[x;\, x_0,\, F_{1,2}(x_0);\, x',\, y']$, which we shall denote by $F^*(x)$, accordingly satisfies the inequalities

$$f(x) > F^*(x) > F_{1,2}(x) \qquad (x_3 \leqq x \leqq x_4). \tag{14}$$

For any set E of numbers dense in (a, b), there are numbers x_5, x_6 in E such that $x_1<x_5<x_3<x_4<x_6<x_2$, for which

$$f(x_5) < F^*(x_5), \qquad f(x_6) < F^*(x_6). \tag{15}$$

Since we have $x_3<(x_5+x_6)/2<x_4$, we find from (14), (15) and Lemma 1 that

$$f\left(\frac{x_5 + x_6}{2}\right) > F^*\left(\frac{x_5 + x_6}{2}\right) > F_{5,6}\left(\frac{x_5 + x_6}{2}\right).$$

Therefore if $f(x)$ is not a sub-$F(x)$ function, there does not exist a set E of numbers dense in (a, b) for which (13) is satisfied.

Actually our method of proof gives more, as stated in the following corollary.

COROLLARY 2. *A necessary and sufficient condition that a function $f(x)$ continuous in (a, b) be a sub-$F(x)$ function is that there exist a positive number ϵ*

and a set E of numbers dense in (a, b) such that for each pair of numbers x_1, x_2 in E satisfying $a<x_1<x_2<b$, there is an x_3 in E such that

$$x_1 + \epsilon(x_2 - x_1) < x_3 < x_2 - \epsilon(x_2 - x_1),$$

for which $f(x_3) \leqq F_{1,2}(x_3)$.

As stated in Corollary 1, if the function $f(x)$ is continuous in (a, b) and if for each pair of numbers x_1, x_2 satisfying $a<x_1<x_2<b$ there is an x_3 satisfying $x_1<x_3<x_2$, such that $f(x_3) \leqq F_{1,2}(x_3)$, then $f(x)$ necessarily is a sub-$F(x)$ function. However, there does exist a function $f(x)$ continuous in (a, b), and a set E dense in (a, b), such that for each pair of numbers x_1, x_2 in E satisfying $a<x_1<x_2<b$, there is an x_3 in E satisfying $x_1<x_3<x_2$, such that $f(x_3) \leqq F_{1,2}(x_3)$, and such that $f(x)$ is not a sub-$F(x)$ function. For let $(a, b) = (0, 1)$, and define $f(x)$ as follows:

$$f(x) = 1/2 \qquad (1/3 \leqq x \leqq 2/3),$$
$$f(x) = 1/4 \qquad (1/9 \leqq x \leqq 2/9),$$
$$f(x) = 3/4 \qquad (7/9 \leqq x \leqq 8/9),$$
$$\cdots\cdots \qquad \cdots\cdots\cdots,$$
$$f(x) \text{ continuous in } (0, 1).$$

If E is the set of all numbers x satisfying

$$1/3 < x < 2/3, \text{ or } 1/9 < x < 2/9, \text{ or } 7/9 < x < 8/9, \text{ or } \cdots,$$

and $\{F(x)\}$ is the set of all nonvertical straight line segments in the strip $0<x<1$, then $f(x)$ satisfies the conditions set forth above.

THEOREM 1. *If $f(x)$ is a sub-$F(x)$ function in (a, b), then for any fixed x_1, x_2 satisfying $a<x_1<x_2<b$, and for all ρ in (a, b) dividing (x_1, x_2) rationally (internally or externally), the points $[\rho, f(\rho)]$ lie on the graph of a uniquely determined continuous sub-$F(x)$ function.*

Proof. For fixed x_1, x_2 with $a<x_1<x_2<b$, let $F_s(x)$ be a member of $\{F(x)\}$ satisfying $F_s(x_1) \geqq f(x_1)$, $F_s(x_2) \geqq f(x_2)$; then for all non-negative integers m, n such that $m/2^n \leqq 1$, we have

$$f\left[x_1 + \frac{m}{2^n}(x_2 - x_1)\right] \leqq F_s\left[x_1 + \frac{m}{2^n}(x_2 - x_1)\right]. \tag{16}$$

For, by Lemma 1, we have

$$F_{1,2}(x) \leqq F_s(x) \qquad (x_1 \leqq x \leqq x_2). \tag{17}$$

From (3) and (17) it follows that we have

$$f\left(x_1+\frac{x_2-x_1}{2}\right) \leqq F_{1,2}\left(x_1+\frac{x_2-x_1}{2}\right) \leqq F_s\left(x_1+\frac{x_2-x_1}{2}\right).$$

The inequality (16) follows by induction.

Next, we shall show that, for the above x_1, x_2, if $F_r(x)$ is a member of $\{F(x)\}$ satisfying

$$F_r(x_1) \geqq f(x_1), \qquad F_r(x_2) \leqq f(x_2),$$

then for all positive integers m, n such that

$$\frac{m}{2^n} \geqq 1, \qquad x_1 + \frac{m}{2^n}(x_2 - x_1) < b,$$

we have

$$f\left[x_1+\frac{m}{2^n}(x_2-x_1)\right] \geqq F_r\left[x_1+\frac{m}{2^n}(x_2-x_1)\right]. \tag{18}$$

From Lemma 1 we obtain

$$F_r(x) \leqq F_{1,2}(x) \qquad (x_2 \leqq x < b). \tag{19}$$

For any positive integers m, n such that $1 \leqq m/2^n \leqq 2$, there is a non-negative integer m' such that $(m+m')/2^{n+1}=1$. Since x_2 is the average of the numbers

$$x_1 + \frac{m'}{2^n}(x_2 - x_1), \qquad x_1 + \frac{m}{2^n}(x_2 - x_1),$$

and since from (16) we have

$$f\left[x_1+\frac{m'}{2^n}(x_2-x_1)\right] \leqq F_{1,2}\left[x_1+\frac{m}{2^n}(x_2-x_1)\right],$$

it follows from (3), (19) and Lemma 1 that if $x_1+m(x_2-x_1)/2^n$ is in (a, b), then

$$f\left[x_1+\frac{m}{2^n}(x_2-x_1)\right] \geqq F_{1,2}\left[x_1+\frac{m}{2^n}(x_2-x_1)\right] \geqq F_r\left[x_1+\frac{m}{2^n}(x_2-x_1)\right].$$

By induction we have (18) also for $m/2^n>2$, provided only that $x_1+m(x_2-x_1)/2^n$ is in (a, b).

Likewise, we find that if $F_t(x)$ is a member of $\{F(x)\}$ satisfying

$$F_t(x_1) \leqq f(x_1), \qquad F_t(x_2) \geqq f(x_2),$$

then for all non-negative integers m, n such that

$$x_1 - \frac{m}{2^n}(x_2 - x_1) > a,$$

we have

$$f\left[x_1 - \frac{m}{2^n}(x_2 - x_1)\right] \geqq F_t\left[x_1 - \frac{m}{2^n}(x_2 - x_1)\right]. \tag{20}$$

Let j be a positive integer sufficiently large so that the number

$$x_0 = x_1 - \frac{1}{2^j}(x_2 - x_1)$$

satisfies $a < x_0 < x_1$. By (16) and (18), for all non-negative integers m, n such that $m/2^n \leqq 1$, we have

$$\begin{aligned} F_{1,2}\left[x_1 + \frac{m}{2^n}(x_2 - x_1)\right] &\geqq f\left[x_1 + \frac{m}{2^n}(x_2 - x_1)\right] \\ &\geqq F_{0,1}\left[x_1 + \frac{m}{2^n}(x_2 - x_1)\right]. \end{aligned} \tag{21}$$

For numbers $x = x_1 + (m/2^n)(x_2 - x_1)$, where m, n are non-negative integers such that $m/2^n \leqq 1$, designate by W the set of points $[x, f(x)]$. The boundedness of W follows from (1) and (21). Since the set of abscissas of points of W is dense in (x_1, x_2), either the closure $\overline{W}$ of W is a continuous curve which is the graph of a single-valued function $y = \overline{W}(x)$ continuous in (x_1, x_2), or else there exists an x_3 satisfying $x_1 < x_3 < x_2$, and numbers y_1, y_2 with $y_1 < y_2$, such that the points with coordinates (x_3, y_1) and (x_3, y_2) belong to $\overline{W}$.

Assume that there do exist such x_3, y_1, y_2. Consider members $F^*(x)$ and $F^{**}(x)$ of $\{F(x)\}$ determined by

$$F^*(x_1) = f(x_1), \qquad F^*(x_3) = F^{**}(x_3) = \frac{y_1 + y_2}{2}, \qquad F^{**}(x_2) = f(x_2).$$

Since (x_3, y_2) is a point of $\overline{W}$, it follows from the continuity of $F^*(x)$ and of $F^{**}(x)$, and from (18) and Lemma 1, that there are positive integers j, k, with $j/2^k < 1$, such that the number $x_4 = x_1 + (j/2^k)(x_2 - x_1)$ is distinct from x_3 and satisfies

$$f(x_4) > F^*(x_4), \qquad f(x_4) > F^{**}(x_4). \tag{22}$$

If $x_4 > x_3$, let

$$x_5 = x_1 + \frac{j+1}{2^k}(x_2 - x_1).$$

From (20) we obtain $f(x) \geqq F_{4,5}(x)$ for all x less than x_4 satisfying

$$x = x_1 + \frac{m}{2^n}(x_2 - x_1),$$

where m, n are positive integers. Since the point (x_3, y_1) of $\overline{W}$ has ordinate $y_1 < F^{**}(x_3)$, it follows from the continuity of $F^{**}(x)$ and of $F_{4,5}(x)$ that for some number x_6 satisfying $x_3 < x_6 < x_4$, we have

$$F^{**}(x_6) = F_{4,5}(x_6). \tag{23}$$

But from (18) and the definition of $F^{**}(x)$, we obtain $F^{**}(x_2) \geqq F_{4,5}(x_2)$, so that there is a number $x_7 > x_4$ such that

$$F^{**}(x_7) = F_{4,5}(x_7). \tag{24}$$

Now (22), (23) and (24) contradict (2), whence the continuity of $\overline{W}(x)$ in the interval $x_1 < x < x_2$ is proved on the assumption that $x_4 > x_3$.

Considering $F^*(x)$ instead of $F^{**}(x)$, we can modify our argument so as to prove $\overline{W}(x)$ continuous in the interval $x_1 < x < x_2$ on the assumption that $x_4 < x_3$.

By considering suitable x_1', x_2', with $a < x_1' < x_1 < x_2 < x_2' < b$, we see that the range of $\overline{W}(x)$ can be extended so that $\overline{W}(x)$ remains continuous in the closed interval $x_1 \leqq x \leqq x_2$.

We shall show that for any positive rational number $p/q < 1$ we have

$$f(x) = \overline{W}(x) \qquad \left[x = x_1 + \frac{p}{q}(x_2 - x_1)\right]. \tag{25}$$

Assume that there is a rational number p_0/q_0 with $0 < p_0/q_0 < 1$, such that

$$f(x_0) \neq \overline{W}(x_0) \qquad \left[x_0 = x_1 + \frac{p_0}{q_0}(x_2 - x_1)\right]. \tag{26}$$

By the method of the last paragraph, there is a function $G(x)$ continuous in the interval $x_1 \leqq x \leqq x_0$ such that for all non-negative integers m, n with $m/2^n \leqq 1$ we have

$$f(x) = G(x) \qquad \left[x = x_1 + \frac{m}{2^n}(x_0 - x_1)\right]. \tag{27}$$

From the continuity of $\overline{W}(x)$ and of $G(x)$, and from (26) and (27), it follows that there is a positive h such that

$$G(x) \neq \overline{W}(x) \qquad (x_0 - h < x \leqq x_0). \tag{28}$$

There exist positive integers m, n, m', n' such that

$$\frac{m}{2^n} < 1, \qquad \frac{m}{2^n}\left(\frac{p_0}{q_0}\right) = \frac{m'}{2^{n'}}, \qquad \frac{m}{2^n}(x_0 - x_1) > x_0 - x_1 - h.$$

But for

$$x_8 = x_1 + \frac{m'}{2^{n'}}(x_2 - x_1) = x_1 + \frac{m}{2^n}(x_0 - x_1),$$

we have

$$x_0 - h < x_1 + \frac{m}{2^n}(x_0 - x_1) = x_8 < x_0 \tag{29}$$

and

$$G(x_8) = f(x_8) = \overline{W}(x_8). \tag{30}$$

Since (29) and (30) contradict (28), (25) follows.

Since $f(x)$ is a sub-$F(x)$ function and since $\overline{W}(x)$ is continuous in the interval $x_1<x<x_2$, it follows from (25) and Lemma 4 that $\overline{W}(x)$ is a sub-$F(x)$ function in (x_1, x_2).

By considering suitable x_1', x_2', with $a<x_1'<x_1<x_2<x_2'<b$, we see that the range of $\overline{W}(x)$ can be extended so that $\overline{W}(x)$ remains continuous in (a, b) and satisfies the conditions of Theorem 1.

4. **Bounded sub-$F(x)$ functions.** We shall use the theorem proved in this section in establishing a more general result in §5.

LEMMA 5. *If $f(x)$ is a sub-$F(x)$ function in the interval (a, b), and is bounded from above in an interval (x_1, x_2) with $a<x_1<x_2<b$, then:*

(i) *For any x_3 with $a<x_3<x_1$ there is an $F^*(x)$ such that $F^*(x_3)=f(x_3)$ and such that $F^*(x)\geqq f(x)$ for all x satisfying $x_3\leqq x\leqq x_2$.*

(ii) *For any x_4 with $x_2<x_4<b$ there is an $F^{**}(x)$ such that $F^{**}(x_4)=f(x_4)$ and such that $F^{**}(x)\geqq f(x)$ for all x satisfying $x_1\leqq x\leqq x_4$.*

Proof. By Lemma 2 there is an $F^*(x)$ satisfying $F^*(x_3)=f(x_3)$, such that $F^*(x)>f(x)$ for all x satisfying $x_1\leqq x\leqq x_2$. For any x' in (x_3, x_1) there is an x'' in (x_1, x_2) such that x' divides (x_3, x'') rationally. Therefore, by Theorem 1 and Lemmas 1 and 3, we have $F^*(x)\geqq f(x)$ for all x satisfying $x_3\leqq x\leqq x_2$.

In like manner, it can be shown that there is an $F^{**}(x)$ satisfying the second part of Lemma 5.

The following corollary follows from Lemma 5 and the continuity of the members of $\{F(x)\}$.

COROLLARY 3. *If $f(x)$ is a sub-$F(x)$ function in the interval (a, b), and is bounded from above in some subinterval of (a, b), then $f(x)$ is bounded from above in every closed subinterval of (a, b).*

THEOREM 2. *If $f(x)$ is a sub-$F(x)$ function in the interval (a, b), and is bounded from above in some subinterval of (a, b), then $f(x)$ is continuous in (a, b).*

Proof. It will be shown that $f(x)$ is continuous at an arbitrary point x_0 of (a, b).

If $I: x_1 \leqq x \leqq x_2$ is a closed subinterval of (a, x_0), then by Corollary 3, $f(x)$ has an upper bound in I. By Lemma 5, there is an $F^*(x)$ such that

$$F^*(x_0) = f(x_0), \qquad F^*(x) \geqq f(x) \qquad (x_1 \leqq x \leqq x_0). \tag{31}$$

For any x' in (x_0, b), there is an x'' in (x_1, x_0) such that x_0 divides (x'', x') rationally. By Theorem 1 and Lemma 3 we have

$$F^*(x) \leqq f(x \qquad (x_0 \leqq x < b). \tag{32}$$

Similarly, there is an x_3 in (x_0, b) and an $F^{**}(x)$ satisfying

$$F^{**}(x_0) = f(x_0), \qquad F^{**}(x) \geqq f(x) \qquad (x_0 \leqq x \leqq x_3), \tag{33}$$

and

$$F^{**}(x) \leqq f(x) \qquad (a < x \leqq x_0). \tag{34}$$

From (31)–(34) and the continuity of the members of $\{F(x)\}$, the continuity of $f(x)$ at x_0 follows. Since x_0 is arbitrary, $a < x_0 < b$, the continuity of $f(x)$ in (a, b) is established.

Corollary 4. *If $f(x)$ is defined in (a, b) and satisfies $f(x) \leqq F_{1,2}(x)$ for all x_1, x_2, x with $a < x_1 < x < x_2 < b$, then $f(x)$ is continuous in (a, b).*

There is a direct proof [1] of Corollary 4 which is simpler than the above proof of Theorem 2.

5. **Measurable sub-$F(x)$ functions.** We shall use Theorem 2 and the following lemma to obtain a result which is more general than Theorem 2.

Lemma 6. *If $f(x)$ is a sub-$F(x)$ function in (a, b) and is bounded from above on a set E of positive measure in (a, b), then $f(x)$ is bounded from above in some subinterval of (a, b).*

Proof. There is a closed subinterval $x_1 \leqq x \leqq x_2$ of (a, b) such that the measure of the part of E in (x_1, x_2) is greater than $2(x_2 - x_1)/3$. Let x_3 and x_4, with $x_3 < x_4$, divide (x_1, x_2) into three equal parts, and let x' be any value satisfying $x_3 \leqq x' \leqq x_4$. Since the measure of the part of E in (x_1, x_2) is greater than $2(x_2 - x_1)/3$, it follows that the measure of the part of E in

$$[x' - (x_4 - x_3), \; x' + (x_4 - x_3)]$$

is greater than $x_4 - x_3$; consequently, there is a $\delta(x')$ less than $x_4 - x_3$ such that $x' - \delta(x')$ and $x' + \delta(x')$ both belong to E.

By assumption there is an M such that we have $f(x) < M$ for all x in E. By Lemma 2, there is an $F^*(x)$ such that $F^*(x) > M$ for all x satisfying $x_1 \leqq x \leqq x_2$. From Lemma 1 and the fact that $f(x)$ is a sub-$F(x)$ function, it follows that

$$F^*(x) > F[x; x - \delta(x), f(x - \delta(x)); x + \delta(x), f(x + \delta(x))] \geqq f(x) \qquad (x_3 \leqq x \leqq x_4).$$

Since $F^*(x)$ is continuous, we conclude that $f(x)$ is bounded, $x_3 \leqq x \leqq x_4$.

THEOREM 3. *If $f(x)$ is a sub-$F(x)$ function in (a, b), and is bounded from above on a set E of positive interior measure in (a, b), then $f(x)$ is continuous in (a, b).*

Proof. Let E_0 be a closed subset of E having positive measure. By Lemma 6 there is a subinterval of (a, b) on which $f(x)$ is bounded from above, so that the continuity of $f(x)$ in (a, b) follows from Theorem 2.

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